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RAINBOW NUMBERS WITH INDEPENDENT CYCLES IN $K_{m,n}$ DEPENDING ON RAINBOW BIPARTITE GRAPHS*

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Abstract

An edge-colored graph G is called a rainbow graph if all the colors on its edges are distinct. Let Γ be a family graph of an edge-colored graph G such as $\Gamma \subseteq G$. The rainbow graph denoted by $rb(G,\Gamma)$ is related to the anti-Ramsey number $AR(G,\Gamma)$. The anti-Ramsey $AR(G,\Gamma)$, introduced by Erdős et al., is the maximum number of colors in an edge coloring graph of G without rainbow copy of any graph in G. Evidently, $rb(G,\Gamma) = AR(G,\Gamma)+1$, $rb(G,\Gamma)$ is the rainbow number of Γ in any edge coloring graph G.

In this paper, we consider the existence of rainbow number with independent cycles in the complete bipartite graphs, denoted by $K_{m,n}$, order m and n with bipartitions (M, N). For this result, we endeavor to construct the complete bipartite graphs on the multi-graphs without independent cycles. Denote that the rainbow number rb $(K_{m,n}, \Omega_2)$ for $m \geq n \geq 5$. Let Ω_2 denote the family of graphs containing two independent cycles. The rainbow number rb $(K_{m,n}, \Omega_2)$ is the minimum number of colors such that $\Omega_2 \subseteq K_{m,n}$, then any edge coloring of $K_{m,n}$ with at least distinct c colors contains a rainbow copy of Ω_2. Without loss of generality, we obtain the result for any $m \geq n \geq 5$, rb $(K_{m,n}, \Omega_2) = 3m + n - 2$. Finally, we hope the main result will be supported at the fiber optic communications network in real life for our country.

Keywords: rainbow graph, rainbow number, independent cycles, edge-colored graph.

Introduction

Now we currently study edge colorings with rainbow numbers to do our paper completely. Here, first we introduce that an edge-colored graph is a graph that its edges have been colored somehow with distinct colors, we can denote it as rainbow edge-colored graph.

Again, we introduce the rainbow numbers for independent cycles in an edge-colored graph. We studied (Bondy & Murty, 1976) terminology and notation to consider finite and simple graphs. (Bondy & Murty, 1976) is completely fulfilled for us to study the basic points in finite and simple graphs. Let Γ be a family of graphs and G be an edge coloring graph and K_n be a complete graph. To determine the rainbow numbers rb (K_n, Γ) for our conditions, first we studied anti-Ramsey number AR (K_n, Γ). The anti-Ramsey number was introduced by Eros, Simonovits and Sos in the 1970s (Erdős et al., 1973). They showed that these are closely related to *Turan number, ex* (K_n, Γ) which means that takes one edge of each color in an edge coloring of K_n, one can show that AR $(K_n, \Gamma) \leq ex$ (K_n, Γ). Γ consists of a single graph H, we can write AR (K_n, Γ) and ex (K_n, Γ). Here, we studied the anti-Ramsey number for a cycle conjectured in (Erdős et al., 1973) by Erdos et al. that AR $(K_n, C_3) = n - 1$ and then the anti-Ramsey number for cycles, AR (K_n, C_k), was determined for $k \leq 6$ in (Alon, 1983; Erdős et al., 1973). Continuously, we studied that the rainbow numbers for cycles, rb $(K_n, C_3) = n$ in (Chartrand & Zhang, 2009). We studied that rainbow numbers for matching in plane triangulation in (Jendrol et al., 2014).

The rainbow numbers is related to the anti-Ramsey numbers that equivalently, rb $(G, H) = AR$ $(G, H) + 1$. Therefore, in our paper to search rainbow numbers, we make anti-Ramsey number focused. Denote by Ω_k the family of multi-graphs that contain k vertex disjoint

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cycles, vertex disjoint cycles are said to be independent cycles. The family of multi- graph does not belong to Ω_k is denoted by $\overline{\Omega}_k$, it is clear that is just the family of forests. It is proved in (Jin & Li, 2009) that anti-Ramsey number $AR(K_n, \Omega_2) = 2n - 2$ for $n \geq 6$. Using the extremal structures theorem for graphs in $\overline{\Omega}_2$ (Bollobas, 1978), we determine that rainbow number $rb(K_{m,n}, \Omega_2)$ for $m \geq n \geq 5$.

Rainbow Bipartite Graphs without Independent Cycles

2.1 Extremal structures theorem for graphs without independent cycles

First, we present extremal structures for the graphs which do not contain two independent cycles.

Theorem 2.1 (Bollobas, 1978) Let G be a multi-graph without two independent cycles. Suppose that $\delta(G) \geq 3$ and there is no any vertex contained in all the cycles of G. Then one of the following six assertions holds (see Figure 1).

(1) G has three vertices and multiple edges joining every pair of the vertices.

(2) G is a K_4 in which one of the triangles may have multiple edges.

(3) $G \cong K_5$.

(4) G is K_5^- such that some of the edges not adjacent to the missing edge may be multiple edges.

(5) G is a wheel whose spokes may be multiple edges.

(6) G is obtained from $K_{3,p}$ by adding vertices or multiple edges joining vertices in the first class.

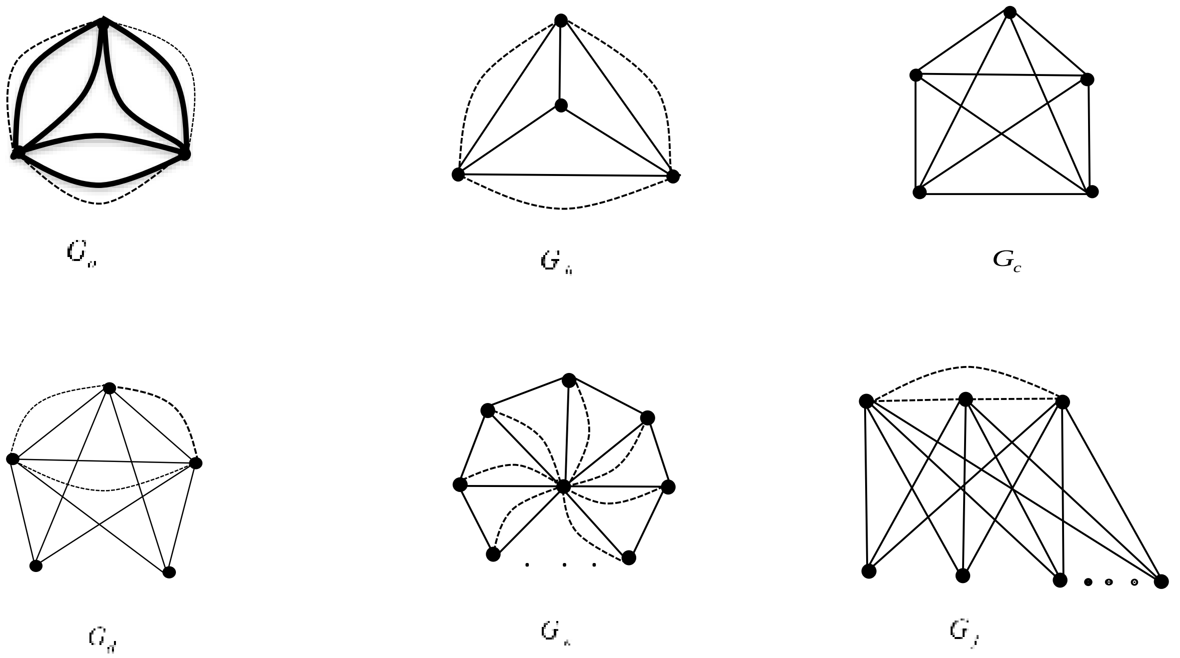

Figure 1: The graphs without independent cycles

Theorem 2.2 (Bollobas, 1978) A multi-graph G does not contain two independent cycles if and only if either it contains a vertex x_0 such that $G-x_0$ is a forest, or it can be obtained from a subdivision G_0 of a graph listed in Figure 1 by adding a forest and at most one edge joining each tree of the forest to G_0.

In general, we have the following result.

Lemma 2.3 Let $m \geq n$ and G be a simple bipartite graph of order m and n with size q without two independent cycles. If G contains a vertex x_0 such that $G-x_0$ is a forest, then $q \leq 2m+2n-3$

By Theorem 2.1, we have the following lemma.

Lemma 2.4 Let $m \geq n$ and G be a simple bipartite graph of size q with m and n vertices in each partite set. Suppose that G can be obtained from a subdivision G_0 of a graph listed in Figure 1 by adding a forest and at most one edge joining each tree of the forest to G_0. Then one of the followings holds.

(1) G_0 is a subdivision of G_a , and then $q \leq 2m+n-2$.

(2) G_0 is a subdivision of G_b , and then $q \leq 2m+n-1$.

(3) G_0 is a subdivision of G_c , and then $q \leq m+n+5$.

(4) G_0 is a subdivision of G_d , and then $q \leq 2m+n$.

(5) G_0 is a subdivision of G_e, and then $q \leq 2m+n-2$.

(6) G_0 is a subdivision of G_f , and then $q \leq 3m+n-3$. Furthermore, the equality holds if and only if G can be obtained from $K_{3,p}$ by adding edges or the multiple edges joining vertices in the first class.

Proof: We prove the lemma by the induction on $m + n$. When $m+n \leq 5$, the lemma holds obviously. Assume that the lemma holds for the graph of order less than $m + n$. Now consider the graph G. Let $G = (M, N: E)$ with $|M| = m$, $|N| = n$ and then $m \geq n$.

Case 1. G_0 is a subdivision of G_a .

Suppose that G_a is a multi-graph containing three vertices such as u, v, w. Take a vertex $x \in G$. If $\delta(G) \leq 1$, then $|V(G-x)| = m+n-1$. Let $\delta(G)=0$. Considering the cases $x \in M$ or $x \in N$, we have

$$|E(G)| = |E(G-x)| \leq \max\{2(m-1)+n-2,\ 2m+(n-1)-2\}$$
$$= 2m+n-3 < 2m+n-2,$$

as desired.

If $\delta(G)=1$, by the same reason we have

$$|E(G)| = |E(G-x)| + 1 \leq \max\{2(m-1)+n-2+1,\ 2m+(n-1)-2+1\}$$
$$= 2m+n-2,$$

as desired again.

So let $\delta(G) \geq 2$. By the Theorem 2.2 and then since G_a is a multi-graph and $\delta(G) \geq 2$, all vertices of $G-\{u,v,w\}$ lie on (multi)-edges between *u*, *v*, *w*. Now, considering the possible vertices belonging to *M* and *N*, we distinguish the following subcases.

Subcase 1.1 $u,v,w \in N$.

There are at most *m* (multi)-edges in G_0. Here, each vertex of *M* is of degree two. Hence, we have that $q \leq 2m$ and we are done.

Subcase 1.2 $u \in M$ and $v,w \in N$.

There are at most *m* – 1 (multi)-edges in G_0. Here, *u* is degree of *n* and each vertex of *M* – *u* is of degree two. Hence, we obtain that $q \leq 2(m-1)+n = 2m+n-2$ and we are done.

Subcase 1.3 $u,v \in M$ and $w \in N$.

Here, there are at most *n* – 1 (multi)-edges in G_0. Here, *w* is degree of *m* and each vertex of $N-w$ is of degree two. Hence, we obtain that $q \leq 2(n-1)+m = 2n+m-2$ and we are done.

Subcase 1.4 $u,v,w \in M$.

There are at most *n* (multi)-edges in G_0. Here, each vertex of *N* is of degree two. Hence, we can deduce that $q \leq 2n$ and we are done. /

Furthermore, these cases such as **Case 2.** G_0 is a subdivision of G_b, **Case 3.** G_0 is a subdivision of G_c, **Case 4.** G_0 is a subdivision of G_d, and **Case 5.** G_0 is a subdivision of G_e can be proved by the similar analysis as above and we omit the details.

Then, we continue to prove the following.

Case 6. G_0 is a subdivision of G_f.

Given that $G_f \cong K_{3,p}$ by adding edges or the (multi)-edges joining vertices in the first class. To say $G_f \subseteq G$. Denote by $G_f = (U,V:E)$. So, let *U* = {*u*, *v*, *w*} be the first class of G_f and then $V = \{x_1, x_2, ..., x_p\}$ be the second class of G_f. Take a vertex $x \in G$. If $\delta(G) = 0$, then $|V(G-x)| = m+n-1$. Considering the cases $x \in M$ or $x \in N$.

$$|E(G)| = |E(G-x)| \leq \max\{3(m-1)+n-3,\ 3m+(n-1)-3\} = 3m+n-4 < 3m+n-3,$$

as desired.

If $\delta(G) = 1$, by the same reason we have

$$|E(G)| = |E(G-x)| + 1 \leq \max\{3(m-1)+n-3+1, 3m+(n-1)-3+1\} = 3m+n-3,$$

as desired again.

So let $\delta(G) \geq 2$. Since G_f is a multi-graph and $\delta(G) \geq 2$, all vertices of $G \setminus (U,V)$ lie on (multi)-edges among the vertices of *U*. Now, considering the possible vertices *u*, *v*, *w* belonging to *M* and *N*, it is enough reason to provide confirmation. So, we distinguish the following subcases.

Subcase 6.1 $u, v, w \in N$

Here, if $V \subseteq M$, there are at most $m - p$ (multi)-edges in G_0. It is obviously that there are at least one vertex of $M \setminus N$ on any (multi)-edges. Here, we need to consider the number of vertices of *p*, since *G* is a bipartite graph,

$$\sum_{x \in M} d(x) = \sum_{y \in N} d(y)$$

$$2(m-p) + 3p \geq 2(n-3) + 3p\,.$$

Therefore, $m \geq n + p - 3,\ 3 \leq p \leq n \leq m$.

Continuously, in details we determine the number of vertices in *V*, so let $V = V_x \bigcup V_{p-x}$ such that *x*-vertices in V_x and $0 \leq x \leq p$. Now, we consider the possible vertices in V_x and V_{p-x} belonging to *M* and *N* so we again distinguish the following sub-subcases.

Sub-subcase 6.1.1 $V_x \subseteq N$ and $V_{p-x} \subseteq M$.

Here, we have at most $m - p + x$ (multi)-edges in G_0. Each vertex of $M \setminus V_{p-x}$ is of degree two and each vertex of V_{p-x} is three. Hence, we have that

$$q \leq 2(m - (p - x)) + 3(p - x) = 2m + p - x \leq 2m + n \text{ (since } 0 \leq x \leq p \leq n)$$

and done.

Sub-subcase 6.1.2 $V_x \subseteq M$ and $V_{p-x} \subseteq N$.

Here we have at most $m - x$ (multi)-edges in G_0. Each vertex of $M \setminus V_x$ is of degree two and each vertex of V_x is three. Hence, we have that

$$q \leq 2(m - x) + 3x \leq 2m + p \leq 2m + n$$

and done.

Here, if $V \subseteq N$, there are at most *m* (multi)-edges in G_0 and each vertex of *M* is of degree two. Hence, we have that $q \leq 2m$ and done.

Subcase 6.2 $u, v, w \in M$.

Here, if $V \subseteq N$, there are at most $n - p$ (multi)-edges in G_0. It is obviously that there are at least one vertex of $N \setminus V_p$ on any (multi)-edges. Here, we need to consider the number of vertices of *p*. Since *G* is a bipartite graph,

$$\sum_{x \in M} d(x) = \sum_{y \in N} d(y)$$

$$2(m - 3) + 3p \leq 2(n - p) + 3p.$$

Since $m \geq n$, so $m \leq n - p + 3,\ p \leq 2$.

Here, also we distinguish in details the number of vertices in *V* by the same reason as above subcase, so we have the following sub-subcases.

Sub-subcase 6.2.1 $V_x \subseteq N$ and $V_{p-x} \subseteq M$.

Here, we have at most $n - x$ (multi)-edges in G_0. Each vertex of $N \setminus V_x$ is of degree two and each vertex of V_x is three. Hence, we have that

$$q \le 2(n-x) + 3x \le 2n + p$$

and done.

Sub-subcase 6.2.2 $V_x \subseteq M$ and $V_{p-x} \subseteq N$.

Here, we have at most $m - p + x$ (multi)-edges in G_0 . Each vertex of $N \setminus V_{p-x}$ is of degree two and each vertex of V_{p-x} is three. Hence, we have that

$$q \le 2(n - p + x) + 3(p - x) = 2n + p - x \le 2n + p$$

and done.

Here, if $V \subseteq M$, then there are at most n (multi)-edges in G_0 and each vertex of N is of degree two. Hence, we have that $q \le 2n$ and done.

Subcase 6.3 $u, v \in N$ and $w \in M$

Here, if $V \subseteq M$, there are at most $m - p - 1$ (multi)-edges in G_0. By the summing of degree,

$$\sum_{x\in M} d(x) = \sum_{y\in N} d(y)$$

$$2(m-p-1)+n+3p \ge 2(n-2)+2p$$

$$2m \ge n-2+p\,.$$

So, $n+2 \le p \le m-1$

Here, also we distinguish in details the number of vertices in V by the same reason as above subcase. So, we have the following sub-subcases.

Sub-subcase 6.3.1 $V_x \subseteq N$ and $V_{p-x} \subseteq M$.

Here, $0 \le x \le p$, we have at most $m - p + x$ (multi-)edges in G_0. Each vertex of $M \setminus (V_{p-x} + w)$ is of degree two and each vertex of V_{p-x} is three and w is of n. Hence, we have that

$$q \le 2(m - p + x - 1) + 3(p - x) + n = 2m + n + p - x - 2 \le 3m + n - 3$$

and done.

Sub-subcase 6.3.2 $V_x \subseteq M$ and $V_{p-x} \subseteq N$.

Here, we have at most $m - x - 1$ (multi)-edges in G_0 . Each vertex of $M \setminus (V_x + w)$ is of degree two and each vertex of V_x is three and w is of degree n. Hence, we have that

$$q \le 2(m - x - 1) + 3x + n = 2m + n + x - 2 \le 2m + n + p - 2 \le 3m + n - 3$$

and done.

Here, if $V \subseteq N$, then we obtain the same result as above. So, we omit the details.

Subcase 6.4 $u \in N$ and $v, w \in M$.

Here, if $V \subseteq N$, then there are at most $n - p - 1$ (multi)-edges in G_0. By the sum degree,

$$\sum_{x\in M} d(x) = \sum_{y\in N} d(y)$$

$$2(m-2)+2p \le 2(n-p-1)+m+3p$$

$$2n \ge m-2+p.$$

So, $m+2 \le p \le n-1.$

Here, also we distinguish in details the number of vertices in V by the same reason as above subcase. So, we have the following sub-subcases.

Sub-subcase 6.4.1 $V_x \subseteq N$ and $V_{p-x} \subseteq M$.

Here, $0 \le x \le p$, we have at most $n - x - 1$ (multi)-edges in G_0. Each vertex of $N \setminus (V_x + w)$ is of degree two and each vertex of V_x is three and w is of m. Hence, we have that

$$q \le 2(n-x-1) + m + 3x = 2n + m + p - x - 2 \le 3n + m - 3$$

and done.

Sub-subcase 6.4.2 $V_x \subseteq M$ and $V_{p-x} \subseteq N$.

Here, we have at most $n - p + x - 1$ (multi)-edges in G_0 . Each vertex of $N \setminus (V_{p-x} + w)$ is of degree two and each vertex of V_{p-x} is three and w is of degree m. Hence, we have that

$$q \le 2(n-p+x-1) + 3(p-x) + m = 2n + m + p - x - 2 \le 2n + m + p - 2 \le 3n + m - 3$$

and done.

Here, if $V \subseteq M$, then we obtain the same result as above. So, we omit the details.

This completes the proof.

More precisely, from the lemma above, we have the following theorem.

Rainbow graphs with independent cycles in $K_{m,n}$

Let Ω_2 be a family of the two independent cycles in a given graph G, then an edge coloring c of $K_{m,n}$ is induced by G and assigns one additional color to all the edges of $\overline{G}$. Clearly, G is a spanning subgraph of $K_{m,n}$, an edge coloring of $K_{m,n}$ induced by G has $|E(G)|+1$ colors. Let G be a complete bipartite graph with bi-partitions $M \cup N$ and $|M| = m$ and $|N| = n$, without loss of generality, in the following result we always assume $m \ge n.$

Theorem 3.1 For any $m \ge n \ge 5$, $rb(K_{m,n}, \Omega_2) = 3m + n - 2$.

Proof: We shall prove the theorem by contradiction. In this theorem, we consider the family of graphs with two rainbow independent cycles, i.e., rainbow member of Ω_2 in any complete bipartite graph $K_{m,n}$ with bipartition $M \cup N$ and $|M| = m$ and $|N| = n$, without loss of generality, $m \ge n.$

Now, we consider **the Lower bound**.

Here, we present an edge coloring of $K_{m,n}$ as follows. Take a copy of $K_{3,m}$ in $K_{m,n}$ and color its edges by distinct colors. For each vertex in $N \setminus V(K_{3,m})$, all the edges incident to it are colored by its own color. So, we get edges incident to it are colored by its own color. So, we get a

($3m + n - 3$)-edge coloring of $K_{m,n}$. Clearly, there is not any rainbow graph with two independent cycles. Then $rb(K_{m,n}, \Omega_2) \geq 3m + n - 2$.

Now, we consider **the Upper bound**.

In order to the upper bond, here we only need to show that any ($3m + n - 2$)-edge coloring c of $K_{m,n}$ always contains a rainbow subgraph belonging to the family Ω_2. By contradiction, we assume that $K_{m,n}$ does not contain any rainbow subgraph in Ω_2. Take a rainbow spanning subgraph G of $K_{m,n}$, which is of size ($3m + n - 2$). That is to say that G contains exactly one edge of each color. By the induction hypothesis, $G \notin \Omega_2$.

From Theorem 2.1, Lemma 2.4, we have that G can be obtained from a subdivision G_0 of a graph listed in Figure 1 by adding a forest and at most one edge joining each tree of the forest to G_0. Since $|E(G)| = 3m + n - 2$, from by Lemma 2.4, G is not a subdivision of G_a, G_b, G_c, G_d, G_e and G_f. Clearly, the number of edges of them is less than $3m + n - 2$. Therefore, this is contradiction.

This completes the proof.

Conclusion

In the current paper, we studied different problems in edge colorings. In particular, we studied rainbow numbers in bipartite graphs with multiple edges and minimum color numbers for rainbow graphs with independent cycles in complete bipartite graph.

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References

Alon, N., (1983) "On a conjecture of Erdős, Simonovits and Sos concerning anti-Ramsey theorems" [J], *J.Graph Theory.,* vol.1: pp.91-94.

Axenovich, M., T. Jaing, and A. Kiindgen, (2001) "Bipartite anti-Ramsey numbers of cycles" [J], *J.Graph Theory,* vol.47: pp.9-28.

Babu, J., L.S. Chandran, and K. Vaidynathan, (2015) "Rainbow matchings in strongly edge-colored graphs" [J], *Discrete Math.,* vol.338: pp.1191-1196.

Bollobas, B., (1978) *Extremal Graph Theory* [M]. Academic Press, New York.

Bondy, J. A., and U. S. R. Murty, (1976) *Graph Theory with Applications* [M]. Macmillan, London and Elsevier, New York.

Chartrand, G., and P. Zhang, (2009) *Chromatics Graph Theory,* Chapman and Hall.

Erdős, P., M. Simonovits, and V.T. Sos, (1973) "Anti-Ramsey theorems", Colloq. *Math.Soc. Janos Bolyai.* Vol.10, Infinite and Finte Sets, Keszthelty (Hungary)., pp.657-665.

Jendrol, S., I. Sciermeyer, and J.H. Tu, (2014) "Rainbow numbers of matchings in plane triangulations" [J], *Discrete Math.,* vol. 331(28): pp. 158-164.

Jin, Z.M., and X.L. Li, (2009) "Anti-Ramsey numbers for graphs with independent cycles"[J], Electron. *J.Combin.,* 16: #R85.